

AD-A154 888

THE OPTIMALITY OF LOWER CONFIDENCE LIMITS FOR THE
RELIABILITY OF SERIES S. (U) WISCONSIN UNIV-MADISON
MATHEMATICS RESEARCH CENTER B HARRIS ET AL. MAR 85

1/1

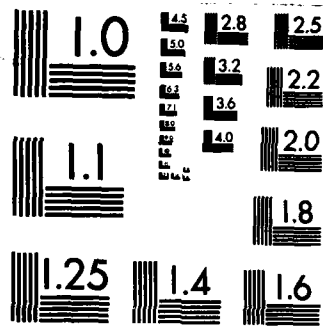
UNCLASSIFIED

MRC-TSR-2883 DAGG29-88-C-0041

F/G 14/4

NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD-A154 880

MRC Technical Summary Report #2803

THE OPTIMALITY OF LOWER CONFIDENCE
LIMITS FOR THE RELIABILITY OF SERIES
SYSTEMS OBTAINED BY THE METHOD OF KEY
TEST RESULTS OR OTHER RELATED
TECHNIQUES

Bernard Harris and Andrew P. Soms

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

March 1985

(Received March 5, 1985)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

DTIC
SELECTE
JUN 11 1985
G
85 06 10 174

2

DTIC FILE COPY

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

THE OPTIMALITY OF LOWER CONFIDENCE LIMITS FOR THE RELIABILITY
OF SERIES SYSTEMS OBTAINED BY THE METHOD OF KEY
TEST RESULTS OR OTHER RELATED TECHNIQUES

Bernard Harris and Andrew P. Soms

Technical Summary Report #2803
March 1985

ABSTRACT

In order to obtain lower confidence limits for the reliability of series systems using binomial subsystem data, K. A. Weaver introduced the method of "key test results". This work was extended by A. Winterbottom. In the present paper, conditions are obtained under which the "method of key test results" gives Buehler optimal lower confidence limits identical with those given by the ordering induced by the maximum likelihood estimator.

AMS (MOS) Subject Classifications: 62N05, 62F25

Key Words: System reliability, Reliability estimation, Key test results, Buehler optimality.

Work Unit Number 4 (Statistics and Probability)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

In this paper, conditions are established under which the total number of defects may be used with no loss in obtaining the lower confidence limit for the reliability of a series system. In such a case, the computations are reduced to the elementary task of determining a lower confidence limit for a single binomial parameter.

Accession For	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By _____	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A/	

DTIC
COPY
INSPECTED
1

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

THE OPTIMALITY OF LOWER CONFIDENCE LIMITS FOR THE RELIABILITY OF SERIES SYSTEMS
OBTAINED BY THE METHOD OF KEY TEST RESULTS OR OTHER RELATED TECHNIQUES

Bernard Harris and Andrew P. Soms

1. Introduction. Let Y_1, Y_2, \dots, Y_k , $k \geq 2$ be independent binomial random variables with parameters (n_i, p_i) , $i = 1, 2, \dots, k$. The objective is to obtain a $1 - \alpha$ lower confidence limit for $h(\tilde{p}) = \prod_{i=1}^k p_i$.

This problem arises naturally in reliability theory, where $h(\tilde{p})$ is the reliability of a series systems of independent components: The data is obtained from independent binomial subsystem experiments. Thus Y_1, Y_2, \dots, Y_k are the number of times each of the k subsystems functioned in n_1, n_2, \dots, n_k Bernoulli trials respectively. We let $X_i = n_i - Y_i$, $i = 1, 2, \dots, k$, be the number of times each of the k subsystems failed in the experiment.

Let

$$f(\tilde{x}, \tilde{p}) = \prod_{i=1}^k \binom{n_i}{x_i} p_i^{n_i - x_i} (1 - p_i)^{x_i} = P_{\tilde{p}}(\tilde{X} = \tilde{x}) \quad (1.1)$$

and let

$$g(\tilde{x}) = \prod_{i=1}^k (n_i - x_i) \quad (1.2)$$

and let

$$\hat{h}(\tilde{p}) = g(\tilde{x}) / \prod_{i=1}^k n_i. \quad (1.3)$$

It is easily verified that $\hat{h}(\tilde{p})$ is the maximum likelihood estimator of $h(\tilde{p})$

and the minimum variance unbiased estimator of $h(\tilde{p})$. For this reason, the performance of lower confidence limits based on $g(\tilde{x})$ provides a yardstick against which the performance of other criteria can be assessed. In particular, we call $m(\tilde{x})$ an ordering function if whenever $\tilde{x}^{(1)} < \tilde{x}^{(2)}$, we have $m(\tilde{x}^{(1)}) > m(\tilde{x}^{(2)})$, $\tilde{x}^{(1)} = (x_1^{(1)}, \dots, x_k^{(1)})$, $0 < x_j^{(1)} < n_1$, $\tilde{x}^{(1)} < \tilde{x}^{(2)}$ means $x_j^{(1)} < x_j^{(2)}$, $j = 1, 2, \dots, k$. It is clear that $g(\tilde{x})$ is an ordering function. Another ordering function of interest to this discussion is $s(\tilde{x}) = \sum_{i=1}^k (n_i - x_i)$. In the discussion that follows, it will be convenient to index the subsystems so that $n_1 < n_2 < \dots < n_k$. This will entail no loss of generality.

For any ordering function $m(\tilde{x})$, the optimal $1 - \alpha$ lower confidence limit determined by $m(\tilde{x})$ is

$$a_{m(\tilde{x}_0)} = \inf_{\tilde{p}} \{h(\tilde{p}) | P_{\tilde{p}}\{m(\tilde{X}) > m(\tilde{x}_0)\} = \alpha\}, \quad (1.4)$$

where \tilde{x}_0 denotes the observed outcome.

An experimental outcome is said to be a "key test result" (A. K. Weaver [2]) if $x_2 = n_2, \dots, x_k = n_k$. We examine Weaver's analysis of key test results in Section 2. A. Winterbottom's [3] extension of Weaver's work is then discussed. We also discuss some other techniques which depends only on the total number of failures.

In Section 3, we present the main points of the paper. There we obtain conditions under which the optimal lower confidence limit for $h(\tilde{p})$ using an ordering function depending on the total number of failures coincides with that given by $g(\tilde{x})$. Correspondingly, we also obtain conditions under which the two procedures differ. These conclusions are compared with the results obtained by Winterbottom and others.

2. The Method of Key Test Results and Other Techniques Based on the Number of Failures. Weaver [2] introduced the notion of a "key test result". He studied the analysis of key test results for $n_1 = n_2 = \dots = n_k = n$. In order to motivate the subject matter of the next section, we summarize his results.

To facilitate obtaining a solution, Weaver replaced (1.4) by

$$a'_{s(\tilde{x}_0)} = \inf_{\tilde{p} \in P^*} \{h(\tilde{p}) | P_{\tilde{p}}\{s(\tilde{X}) > s(\tilde{x}_0)\} = \alpha\}, \quad (2.1)$$

where

$$P^* = \{\tilde{p} : p_1 = p_2 = \dots = p_j = \rho, p_{j+1} = p_{j+2} = \dots = p_k = 1, 1 < j < k\}.$$

Then

$$P_{\tilde{p}}\{s(\tilde{X}) > s(\tilde{x}_0)\} = \sum \prod_{i=1}^j \binom{n}{x_i} \rho^{n-x_i} (1-\rho)^{x_i}, \quad (2.2)$$

where the sum is over all x_1, x_2, \dots, x_j such that $kn - \sum_{i=1}^j x_i > s(\tilde{x}_0)$.

For $\tilde{p} \in P^*$, $h(\tilde{p}) = \rho^j$. In addition, since $p_1 = p_2 = \dots = p_j = \rho$, we can write

$$P\left\{\sum_{i=1}^k x_i = z\right\} = \binom{jn}{z} \rho^{jn-z} (1-\rho)^z$$

and thus

$$\begin{aligned} P\{kn - \sum_{i=1}^j x_i > s(\tilde{x}_0)\} &= P\left\{\sum_{i=1}^k x_i < kn - s(\tilde{x}_0)\right\} \\ &= \sum_{z=0}^{kn-s(\tilde{x}_0)} \binom{jn}{z} \rho^{jn-z} (1-\rho)^z. \end{aligned} \quad (2.3)$$

For fixed j , the equation

$$f(j, \rho) = \sum_{z=0}^{kn-s(\tilde{x}_0)} \binom{jn}{z} \rho^{jn-z} (1-\rho)^z = \alpha, \quad 0 < \alpha < 1,$$

has a unique solution in ρ , which we denote by $\rho_j(\alpha)$. Thus, to obtain the infimum required in (2.1), it suffices to set $\rho = \rho_j(\alpha)$ in (2.3), then

$$a'_{s(\tilde{x}_0)} = \min\{\rho_1(\alpha), (\rho_2(\alpha))^2, \dots, (\rho_k(\alpha))^k\}. \quad (2.4)$$

Weaver gave some heuristic arguments which suggest that for $\rho_j(\alpha)$ sufficiently close to unity for all j , $a'_{s(\tilde{x}_0)} = \rho_1(\alpha)$. This conclusion appears to hold with much greater generality than required by Weaver. This impression is based on extensive numerical experimentation. If that conclusion is assumed valid, then Weaver's method is an extraordinarily simple method of obtaining lower confidence limits for the reliability of series systems of independent subsystems. Consequently, it seems desirable to investigate its properties.

Weaver had a number of additional suggestions, but without proofs. In actuality he proposed a more complicated ordering function than $s(\tilde{x})$, but he did not make substantive use of this ordering function. For experimental outcomes more general than key test results, he proposed comparing

$$\left\{ \sum_{0}^{s(x)-1} \binom{n}{x} \rho_1^{n-x} (1 - \rho_1)^x, \sum_{0}^{s(x)} \binom{jn}{x} (\rho_j)^{jn-x} (1 - \rho_j)^x, j = 2, 3, \dots, k \right\}.$$

Each of the k expressions above are equated to α and the minimum of $\rho_1(\alpha), \rho_2^2(\alpha), \dots, \rho_k^k(\alpha)$ is taken as the lower confidence limit. Note that $\rho_1(\alpha)$ uses $s(x) - 1$ as the upper level of summation. For unequal sample sizes, Weaver suggested replacing each x_1 by $n_1 x_1 / n_1$. Weaver notes that this should provide a conservative result. (Intuitively this would seem to be the case).

The use of $s(\tilde{x})$ as an ordering function has a long history. It is suggested by the following considerations. From (1.3)

$$\hat{h}(p) = \frac{\prod_{i=1}^k (n_i - x_i)}{\prod_{i=1}^k n_i}.$$

Assume that p_1, p_2, \dots, p_k are all "close to unity", so that the x_i , $i = 1, 2, \dots, k$ assume "small values" with probability "close to unity". Also assume that n_1, n_2, \dots, n_k are all large. Then, approximate as follows:

$$\hat{h}(p) = \frac{\prod_{i=1}^k (1 - \frac{x_i}{n_i})}{\prod_{i=1}^k 1} \sim 1 - \sum_{i=1}^k \frac{x_i}{n_i}. \quad (2.5)$$

If the n_i , $i = 1, 2, \dots, k$ are "approximately equal", one replaces n_i by an average value, say \bar{n} . Thus,

$$\hat{h}(p) \sim 1 - \sum_{i=1}^k x_i / \bar{n} \quad (2.6)$$

and the statistic $\hat{h}(p)$ depends only on $\sum_{i=1}^k x_i$ and thus is statistically equivalent to $s(\tilde{X})$.

Methods using the approximations (2.5) and (2.6) are discussed in I. V. Pavlov [1].

Winterbottom [3] studied the use of key test results. He proposed basing the lower confidence limit on the key test results $([n_1 \hat{h}(p)], n_2, \dots, n_k)$, $([n_1 \hat{h}(p)] + 1, n_2, \dots, n_k)$, $n_1 < n_2 < \dots < n_k$. However, in the theoretical appendix to the paper, he studies conditions under which key test results provide the same lower confidence limit as the optimal lower confidence limit based on $\hat{h}(p)$. In this instance, he assumes $n_1 = n_2 = \dots = n_k = n$ and utilizes the condition that $\sum_{i=1}^k x_i < x_{1,0}$, where $x_{1,0}$ is the observed number of failures on the first component (since this is a key test result).

Specifically, Winterbottom concludes that this holds whenever $(kn - \sum_{i=1}^k x_i) > s^*$, where

$$s^* = \max_{0 \leq s \leq n} \{s[kn^{k-1}(s-1)]^{1/k} < (k-1)n + s - 2\} . \quad (2.7)$$

If $n_k > n_1$, for key test results, Winterbottom uses the same lower limit that would be obtained if $n_k = n_1$ and asserts that if the lower confidence limit obtained for key test results satisfies (2.7), then this lower confidence limit will also agree with that given by the optimal confidence limit determined by $\hat{h}(p)$.

3. Comparison of Methods Based on the Number of Failures and Methods Based on

the Minimum Variance Unbiased Estimator. In this section, we obtain the

theoretical results which provide the comparison between optimal lower confidence

limits based on the two different procedures discussed in the previous sections.

To establish the principal result, several preliminary lemmas are introduced.

Lemma 3.1. Let $0 < n_1 < n_2 < \dots < n_k$, $k > 2$ and $0 < x_i < n_i$, $i = 1, 2, \dots, k$ be given. Then if $\sum_{i=1}^k x_i$ is fixed,

$$\prod_{i=1}^k (n_i - x_i) > (n_1 - \sum_{i=1}^k x_i) \prod_{i=2}^k n_i . \quad (3.1)$$

Proof. If $\sum_{i=1}^k x_i > n_1$, the conclusion is obtained trivially. Hence assume $\sum_{i=1}^k x_i < n_1$. For $1 < j < k$, since $n_j > n_1$, we have

$$(n_j - n_1)x_j + x_j \sum_{i=1}^{j-1} x_i > 0 . \quad (3.2)$$

Then,

$$x_j \sum_{i=1}^{j-1} x_i + n_1 n_j - n_j \sum_{i=1}^{j-1} x_i - n_1 x_j > n_1 n_j - n_j \sum_{i=1}^{j-1} x_i - n_j x_j ,$$

or

$$(n_1 - \sum_{i=1}^{j-1} x_i)(n_j - x_j) > n_j(n_1 - \sum_{i=1}^j x_i) .$$

Successively setting $j = 2, 3, \dots, k$, we obtain the following inequalities:

$$(n_1 - x_1)(n_2 - x_2) > n_2(n_1 - x_1 - x_2)$$

$$(n_1 - x_1 - x_2)(n_3 - x_3) > n_3(n_1 - x_1 - x_2 - x_3)$$

$$\vdots$$

$$(n_1 - x_1 - x_2 - \dots - x_{k-1})(n_k - x_k) > n_k(n_1 - x_1 - \dots - x_k) . \quad (3.3)$$

The conclusion follows readily upon multiplying the above inequalities.

Lemma 3.2. Let $0 < n_1 < n_2 < \dots < n_k$ and $0 < x_i < n_i$, $i = 1, 2, \dots, k$ with $\sum_{i=1}^k x_i < n_1$ be given. Let $0 < z_i < n_i$, $i = 1, 2, \dots, k$. Then if

$$A = \{z : \sum_{i=1}^k z_i < \sum_{i=1}^k x_i\}, \quad A^* = \{z : \prod_{i=1}^k (n_i - z_i) > \prod_{i=1}^k (n_i - x_i)\} ,$$

a necessary condition that $A = A^*$ is, $x_i = 0$, $i = 1, 2, \dots, k$, or $x_j = \gamma$,

$0 < \gamma < n_1$, $x_i = 0$, $i \neq j$, and $n_j = n_1$.

Proof. Let $B = \{\tilde{z} : \sum_{i=1}^k \tilde{z}_i = \sum_{i=1}^k x_i\}$. If $\tilde{z} \in B \cap A^*$, then, utilizing Lemma 3.1, we have

$$\begin{aligned} \prod_{i=1}^k (n_i - z_i) &> \prod_{i=1}^k (n_i - x_i) > (n_1 - \sum_{i=1}^k x_i) \prod_{i=2}^k n_i \\ &= (n_1 - \sum_{i=1}^k z_i) \prod_{i=2}^k n_i . \end{aligned} \quad (3.4)$$

Let $\tilde{z}^* = (\sum_{i=1}^k x_i, 0, \dots, 0)$. Then $\tilde{z}^* \in B \subset A$ and since $\sum_{i=1}^k x_i < n_1$ was hypothesized, it follows that $0 < z_i^* < n_i$, $i = 1, 2, \dots, k$. Thus,

$$\prod_{i=1}^k (n_i - z_i^*) = (n_1 - \sum_{i=1}^k x_i) \prod_{i=2}^k n_i . \quad (3.5)$$

Assume that $\tilde{z}^* \in A^*$; then

$$(n_1 - \sum_{i=1}^k x_i) \prod_{i=2}^k n_i > \prod_{i=1}^k (n_i - x_i) > (n_1 - \sum_{i=1}^k x_i) \prod_{i=2}^k n_i . \quad (3.6)$$

order that (3.6) holds, equality must hold in each of the inequalities (3.3), equivalently in (3.2). Thus,

$$(n_j - n_1)x_j + x_j \sum_{i=1}^{j-1} x_i = 0, \quad j = 2, \dots, k. \quad (3.7)$$

Since $n_j > n_1$, $0 < x_j$; it follows that

$$(n_j - n_1)x_j = 0, \quad x_j \sum_{i=1}^{j-1} x_i = 0, \quad j = 2, \dots, k. \quad (3.8)$$

Assume that $x_s \neq 0$, $s < k$, then $x_s(x_1 + x_2 + \dots + x_{s-1}) = 0$, which implies that $x_i = 0$, $i < s$. Furthermore, $(n_s - n_1)x_s = 0$ implies $n_s = n_1$ and thus $n_1 = n_2 = \dots = n_s$.

Propollary. If $x_i = 0$, $i = 1, 2, \dots, k$, then $A = A^*$.

Proof. If $\tilde{z} \in A$, then $z_i = 0$, $i = 1, 2, \dots, k$ and $\prod_{i=1}^k (n_i - z_i) = \prod_{i=1}^k n_i = \prod_{i=1}^k (n_i - x_i)$. Thus $\tilde{z} \in A^*$. If $\tilde{z} \in A^*$, $z_i = 0$, $i = 1, 2, \dots, k$ and $\tilde{z} \in A$.

The following discussion shows that the condition of Lemma 3.2 is not

sufficient. If the condition holds, namely $x_j = \gamma$, $0 < \gamma < n_1$, $x_i = 0$, $i \neq j$, $j = n_1$, $1 < j < k$. Then $\sum_{i=1}^k x_i = \gamma$ and $\prod_{i=1}^k (n_i - x_i) = (n_1 - \gamma) \prod_{i=2}^k n_i$. We construct a $\tilde{z} \in A^*$ with \tilde{z} not in A .

Thus let $n_1 < n_2 < \dots < n_k$ be given with $z_1 = z_2 = \dots = z_{k-1} = 0$,

$n_k = \beta > \gamma$. Then

$$\prod_{i=1}^k (n_i - z_i) = (n_1 - \gamma) \prod_{i=2}^k n_i, \quad \prod_{i=1}^k (n_i - z_i) = (n_k - \beta) \prod_{i=1}^{k-1} n_i.$$

In order that $\tilde{z} \in A^*$, we must have

$$(n_k - \beta) \prod_{i=1}^{k-1} n_i > (n_1 - \gamma) \prod_{i=2}^k n_i$$

or

$$(n_k - \beta)n_1 > (n_1 - \gamma)n_k.$$

This holds whenever

$$\beta n_1 < \gamma n_k.$$

However, $\tilde{z} \notin A$ whenever $\beta > \gamma$. Thus, it suffices to set $\beta = \gamma + 1$. Then there is a $\tilde{z} \in A^*$ with $\tilde{z} \notin A$ whenever

$$\frac{\gamma + 1}{\gamma} < \frac{n_k}{n_1}.$$

In particular, for $\gamma = 1$, $A \neq A^*$ if $n_k > 2n_1$ and if $\gamma = 2$, $A \neq A^*$ whenever $n_k > (3/2)n_1$.

Lemma 3.3. Let $0 < n_1 < n_2 < \dots < n_k$, $k > 2$, $0 \leq \gamma < n_1$ be given. Let $\tilde{z} = (z_1, z_2, \dots, z_k)$, $0 \leq z_i < n_i$, $i = 1, 2, \dots, k$. Let $A_\gamma = \{\tilde{z} : \sum_{i=1}^k z_i \leq \gamma\}$, $A_\gamma^* = \{\tilde{z} : \prod_{i=1}^k (n_i - z_i) > (n_1 - \gamma) \prod_{i=2}^k n_i\}$. Then $A_\gamma \subset A_\gamma^*$, $\alpha = 1, 2, \dots, n_1 - 1$.

Proof. If $\gamma = 0$, the conclusion is immediate from the Corollary to Lemma

4.2. Thus if $n_1 = 1$, the lemma is established. Hence, assume $n_1 > 1$.

We proceed by induction. Assume $A_\gamma \subset A_\gamma^*$, $\gamma = 0, 1, \dots, m$, $m < n_1 - 1$. From their respective definitions, it is immediate that

$$A_\gamma \subset A_{\gamma+1}, \quad A_\gamma^* \subset A_{\gamma+1}^*. \quad (3.9)$$

Thus, from the induction hypothesis, it is immediate that

$$A_\gamma \subset A_{\gamma+1}^*. \quad (3.10)$$

Consequently, the conclusion is established if we show that

$$A_{\gamma+1} - A_\gamma \subset A_{\gamma+1}^*.$$

However

$$A_{\gamma+1} - A_{\gamma} = \{ \tilde{z} : \sum_{i=1}^k z_i = \gamma + 1 \} .$$

From Lemma 3.1; if $\tilde{z} \in A_{\gamma+1} - A_{\gamma}$, then

$$\prod_{i=1}^k (n_i - z_i) > (n_1 - \gamma - 1) \prod_{i=2}^k n_i ;$$

consequently such \tilde{z} is in $A_{\gamma+1}^*$, establishing the conclusion.

Lemma 3.4. Let $0 < n_1 < n_2 < \dots < n_k$, $0 < z_i < n_1$, $i = 1, 2, \dots, k$. Let

$A_{\gamma}^C = \{ \tilde{z} : \sum_{i=1}^k z_i > \gamma + 1 \}$ where z_i , $i = 1, 2, \dots, k$ are real valued and $\gamma < n_1$ is a positive integer.

Let j be the least index, $1 < j < k$ such that

$$n_j - \left(\sum_{i=j}^k n_i - \gamma - 1 \right) / (k - j + 1) > 0 . \quad (3.11)$$

Then

$$\max_{\tilde{z} \in A_{\gamma}^C} \prod_{i=1}^k (n_i - z_i) = \left[\frac{(n_j - \gamma) + \sum_{i=j+1}^k n_i - 1}{k - j + 1} \right]^{k-j+1} \prod_{i=1}^{j-1} n_i . \quad (3.12)$$

Proof. It suffices to consider vectors \tilde{z} such that $\sum_{i=1}^k z_i = \gamma + 1$, since, if $\sum_{i=1}^k z_i > \gamma + 1$, reducing any positive component of \tilde{z} will increase

$\prod_{i=1}^k (n_i - z_i)$. Hence, let

$$C = \{ \tilde{z} : \sum_{i=1}^k z_i = \gamma + 1, 0 < z_i < n_1, i = 1, 2, \dots, k \}$$

and let

$$D = \{ \tilde{z} : z_i < n_1, i = 1, 2, \dots, k \} .$$

Clearly C and D are convex sets. Furthermore, on D , $\sum_{i=1}^k \log(n_i - z_i)$ is a strictly concave function. Thus it has a unique maximum on $D^* = \{\tilde{z} : \sum_{i=1}^k z_i = \gamma + 1, z_i < n_i, i = 1, 2, \dots, k\}$, which we now characterize.

Using Lagrange multipliers or alternatively, from the arithmetic mean - geometric mean inequality, we have

$$\max_{\tilde{z} \in D^*} \prod_{i=1}^k (n_i - z_i) = \left(\frac{(n_1 - \gamma) + \sum_{i=2}^k n_i - 1}{k} \right)^k = g(\tilde{n}, \gamma, k). \quad (3.13)$$

The maximum is attained for $\tilde{z}^* = (z_1^*, z_2^*, \dots, z_k^*)$, where

$$z_i^* = n_i - \left(\sum_{i=1}^k n_i - \gamma - 1 \right) / k, \quad i = 1, 2, \dots, k, \quad (3.14)$$

$$z_1^* < z_2^* < \dots < z_k^*.$$

Thus, since $C \subset D^*$,

$$\max_{\tilde{z} \in C} \prod_{i=1}^k (n_i - z_i) < \max_{\tilde{z} \in D^*} \prod_{i=1}^k (n_i - z_i) \quad (3.15)$$

and equality holds whenever \tilde{z}^* is in C . Since $\sum_{i=1}^k z_i^* = \gamma + 1 > 0$, there is a least index i such that $z_i^* > 0$. If \tilde{z}^* is in C^c , then $z_1^* < 0$. From the strict concavity of $\sum_{i=1}^k \log(n_i - z_i)$ and since $C \subset D^*$, $\max_{\tilde{z} \in C} \prod_{i=1}^k (n_i - z_i)$ will be attained on the boundary of C whenever \tilde{z}^* is in C^c . Thus, set $z_1 = 0$ and repeat the above computations with k replaced by $k - 1$, $i = 2, \dots, k$. That is, set $\tilde{z}_1^* = (0, z_{21}^*, \dots, z_{k1}^*)$, where

$$z_{i1}^* = n_i - \left(\sum_{i=2}^k n_i - \gamma - 1 \right) / (k - 1), \quad i = 2, \dots, k, \quad (3.16)$$

$$g_1(\tilde{n}, \gamma, k) = n_1 \prod_{i=2}^k (n_i - z_{i1}^*) = n_1 \left(\frac{\sum_{i=2}^k n_i - \gamma - 1}{k-1} \right)^{k-1}. \quad (3.17)$$

If $z_{21}^* < 0$, repeat the process as indicated above. This is to be continued until the solution is in C . It can be easily seen that the process terminates, for if $z_1^*, z_{21}^*, \dots, z_{k-1, k-2}^*$ are all negative, then $\tilde{z}_k^* = (0, 0, \dots, 0, z_{k, k-1}^*)$ and $z_{k, k-1}^* > 0$ since $\sum_{i=1}^k z_i = \gamma + 1$ implies $z_{k, k-1}^* = \gamma + 1 > 0$. The conclusion follows immediately and is given by (3.11) and (3.12).

Lemma 3.5. Let $0 < n_1 < n_2 < \dots < n_k$, $0 < z_i < n_i$, $i = 1, 2, \dots, k$. Let $\gamma < n_1$ be a non-negative integer. Let

$$A_\gamma = \{\tilde{z} : \sum_{i=1}^k z_i < \gamma\}, \quad A_\gamma^* = \{z : \prod_{i=1}^k (n_i - z_i) > (n_1 - \gamma) \prod_{i=2}^k n_i\}. \quad (3.18)$$

Then if $A_\gamma^* \subset A_\gamma$, we have $A_{\gamma-1}^* \subset A_{\gamma-1}$.

Proof. If $\gamma = 0$, the lemma is trivially true. Hence assume $\gamma > 0$. Now

$A_\gamma^* \subset A_\gamma$ is equivalent to $A_\gamma^C \subset A_\gamma^{*C}$, or whenever $\sum_{i=1}^k z_i > \gamma$ it follows that $\prod_{i=1}^k (n_i - z_i) < (n_1 - \gamma) \prod_{i=2}^k n_i$. Let $\tilde{z} \in A_\gamma^C$ and assume $\tilde{z} \in A_\gamma^{*C}$. Then there is a z_j such that $z_j > 0$. Let $\tilde{z}_1 = (z_1, z_2, \dots, z_j - 1, \dots, z_k)$. Then $\tilde{z}_1 \in A_{\gamma-1}^C$ and

$$\prod_{i=1}^k (n_i - z_{i1}) = \prod_{i=1}^k (n_i - z_i) + \prod_{i \neq j} (n_i - z_i) < (n_1 - \gamma) \prod_{i=2}^k n_i + \prod_{i \neq j} n_i.$$

If $j = 1$,

$$(n_1 - \gamma) \prod_{i=2}^k n_i + \prod_{i=2}^k n_i = (n_1 - \gamma + 1) \prod_{i=2}^k n_i$$

and $\tilde{z}_1 \in A_{\gamma-1}^{*C}$.

Similarly, if $j \neq 1$, then

$$\prod_{i \neq j} n_i < \prod_{i=2}^k n_i$$

upon replacing n_1 by n_j ; and again $\tilde{z}_1 \in A_{\gamma-1}^*$.

Combining the preceding lemmas, we can establish the following theorems.

Theorem 3.1. Let $0 < n_1 < n_2 < \dots < n_k$, $k > 2$, $0 < x_i < n_i$, $i = 1, 2, \dots, k$ be given with $\sum_{i=1}^n x_i < n_1$. Let $\tilde{z} = (z_1, z_2, \dots, z_k)$, where $0 < z_i < n_i$, $i = 1, 2, \dots, k$. Let $A = \{\tilde{z} : \sum_{i=1}^k z_i < \sum_{i=1}^k x_i\}$,

$$A^* = \{z : \prod_{i=1}^k (n_i - z_i) > \prod_{i=1}^k (n_i - x_i)\}.$$

Then $A = A^*$ if and only if $x_j = \gamma$, $0 < \gamma < n_1$, $x_i = 0$, $i \neq j$, $n_j = n_1$ and

$$\max_{\tilde{z} \in A} \prod_{i=1}^k (n_i - z_i) < (n_1 - \gamma) \prod_{i=2}^k n_i. \quad (3.19)$$

Proof. The necessity is immediate from Lemma 3.2. From Lemma 3.3, we then have that $A \subset A^*$. From (3.19), it follows immediately that $A^* \subset A$, establishing the conclusion:

Theorem 3.2. Let $0 < n_1 < n_2 < \dots < n_k$, $k > 2$, $0 < x_i < n_i$, $i = 1, 2, \dots, k$ be given, where $\sum_{i=1}^k x_i < n_1$. Then a sufficient condition that the Buehler optimal lower confidence limit obtained by the method of key test results coincides with that given by the use of the maximum likelihood estimator as an ordering function is

$$\left(\frac{(n_j - \gamma) + \sum_{i=j+1}^k n_i - 1}{k - j + 1} \right)^{k-j+1} \prod_{i=1}^{j-1} n_i < (n_1 - \gamma) \prod_{i=2}^k n_i, \quad (3.20)$$

where j is specified by Lemma 3.4. In this case, the method of key test results is valid for all $x_1 < \gamma$, where γ satisfies (3.20).

Then

$$P_p\{g(\tilde{X}) > g(\tilde{x}_0)\} < I_k \prod_{i=1}^k P_i (n_1 - x_1, x_1 + 1), \quad (3.21)$$

where

$$I_p(n - t, t + 1) = \frac{1}{B(n - t, t + 1)} \int_0^p u^{n-t-1} (1 - u)^t du = \sum_{i=0}^t \binom{n}{i} p^{n-i} q^i. \quad (3.22)$$

Proof. The proof is an immediate consequence of Lemmas 3.4 and 3.5 and the following observations. The set $A = \{z : \sum_{i=1}^k z_i < \sum x_i\}$ is the set whose probability is sought in (1.4), when the method of key test results is utilized and the set $A^* = \{z : \prod_{i=1}^k (n_i - z_i) > \prod_{i=1}^k (n_i - x_i)\}$ is the corresponding set when $m(x) = g(x)$.

Also,

$$\max_{z \in A} \prod_{i=1}^k (n_i - z_i) < \left(\frac{(n_j - \gamma) + \left(\sum_{i=j+1}^k n_i - 1 \right)}{k - j + 1} \right)^{k-j+1} \prod_{i=1}^{j-1} n_i < (n_1 - \gamma) \prod_{i=2}^k n_i,$$

j as specified in (3.12), since the maximum as given in Lemma 3.4 was calculated over a set of real numbers and the application requires that $x_1, x_2, \dots, x_k, z_1, z_2, \dots, z_k$ be integer valued.

Remark. Some comments are required in applying Lemma 3.4. The solutions are real valued. However, the application requires integer values. From non-integral solutions, utilizing the strict concavity, one can easily determine the required integers.

Lemma 3.5 shows that there is a largest γ for which $A_{\beta}^* \subset A_{\beta}$ for all $0 < \beta < \gamma$. If $\gamma = 0$, from the Corollary to Lemma 4.2, we have that $A_Y^* \subset A_Y$. If $\gamma = n_1 - 1$, then for $A_Y^* \subset A_Y$, we must have

$$\prod_{i=1}^k (n_i - z_i) < \prod_{i=2}^k n_i < \left(\sum_{i=2}^k n_{i/k} \right)^k .$$

REFERENCES

- [1] Pavlov, I. V., "A confidence estimate of system reliability from component testing results", Izvestiya Akad. Nauk. Tech. Kibernetiky, 3 (1973), 52-61.
- [2] Weaver, K. A., "The determination of the lower confidence limit of the reliability of a serial system", Proc. of the NATO Conference on Reliability Theory, Turin, 1969, Gordon and Breach, New York, (1971), 335-355.
- [3] Winterbottom, A., "Lower limits for series system reliability from binomial subsystem data", J. Amer. Statist. Assoc., 69 (1974), 782-788.

BH:APS:scr

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2803	2. GOVT ACCESSION NO. AD-115488	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) THE OPTIMALITY OF LOWER CONFIDENCE LIMITS FOR THE RELIABILITY OF SERIES SYSTEMS OBTAINED BY THE METHOD OF KEY TEST RESULTS OR OTHER RELATED TECHNIQUES		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
7. AUTHOR(s) Bernard Harris and Andrew P. Soms		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, North Carolina 27709		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 4 - Statistics and Probability
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE March 1985
		13. NUMBER OF PAGES 17
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) System reliability Reliability estimation Key test results Buehler optimality		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) In order to obtain lower confidence limits for the reliability of series systems using binomial subsystem data, K. A. Weaver introduced the method of "key test results". This work was extended by A. Winterbottom. In the present paper, conditions are obtained under which the "method of key test results" gives Buehler optimal lower confidence limits identical with those given by the ordering induced by the maximum likelihood estimator.		

END

FILMED

7-85

DTIC